

OPERATOR SPLITTING FOR DISSIPATIVE DELAY EQUATIONS

ANDRÁS BÁTKAI, PETRA CSOMÓS, AND BÁLINT FARKAS

ABSTRACT. We investigate operator splitting methods for a special class of nonlinear partial differential equations with delay. Using results from the theory of nonlinear contraction semigroups in Hilbert spaces, we explain the convergence of the splitting procedure. The order of the convergence is also given in some important, linear special cases.

1. INTRODUCTION

Partial differential equations with delay play an increasingly important role in modeling physical, chemical, economical, etc. phenomena, since it is quite natural to assume that past occurrences effect the model. For further motivation see for example the monographs by Wu [26] or Bátkai and Piazzera [3].

There has been lots of work describing the asymptotic behavior and regularity of solutions, as well as in the numerical analysis of ordinary differential equations with delay, see for example the monograph by Bellen and Zennaro [4]. The numerical analysis of partial differential equations with delay, however, seems to be in its infancy. The present paper aims to contribute to this topic in analyzing an operator splitting procedure for nonlinear partial differential equations with delay.

Operator splitting is widely used in numerical analysis of complex systems. The idea is to decompose the differential equation into simpler equations which can be solved in an effective way, and then represent the solution of the original equation using product formulae. For ordinary differential equations, the theory seems to be quite complete, as witnessed in Hairer-Lubich-Wanner [15, Section II.4,5]. There has also been enormous progress in the theoretical investigation of splitting procedures for infinite dimensional systems in recent years, see for example the monographs by Faragó and Havasi [13], Holden et al. [17], Hundsdorfer and Verwer [18], or Lubich [24]. See also the recent papers by Bátkai et al. [2, 1], where nonautonomous equations and spatial approximations are also considered. Unfortunately, abstract results analyzing the order of convergence are rather incomplete, and, as it seems, can be applied to delay equations only with enormous difficulty.

The idea to apply splitting procedures to delay equations is the following. Consider, e.g., a delay equation of the form

$$u'(t) = \Delta u(t) + \sin \left(u(t-1) + \int_{-0.5}^0 \sigma \cdot u(t+\sigma) \, d\sigma \right).$$

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with some initial and boundary conditions. The delay term appearing here represents the two main classes of possible delays in applications: point delays corresponding to dependence on a single event in the past, and distributed delays (given by an integral term) corresponding to dependence on a whole time period in the past. In our opinion, distributed delays are more realistic in modeling.

Since the delay term is in a way a scalar operator, it is natural to decompose the equation into two sub-problems: the heat equation

$$w'(t) = \Delta w(t),$$

and a scalar-valued delay equation

$$v'(t) = \sin \left(v(t-1) + \int_{-0.5}^0 \sigma \cdot v(t+\sigma) d\sigma \right).$$

Both equations can be solved numerically in an effective way. Note that the second equations becomes here an ordinary differential equation with delay, hence the methods described in Bellen and Zennaro [4] can be applied.

Our aim is to put this example in an abstract perspective explaining the convergence and analyzing the order of convergence in some special cases. To be more specific, we will consider on the Hilbert space H the following abstract delay differential equation

$$\begin{cases} \frac{du(t)}{dt} = Bu(t) + \Phi u_t, & t \geq 0, \\ u(0) = x \in H, \\ u_0 = f \in L^p([-1, 0]; H), \end{cases}$$

where $u : [-1, \infty) \rightarrow H$ is unknown and the *history function* u_t defined by $u_t(\sigma) := u(t + \sigma)$ for $s \in [-1, 0]$. The further precise definitions and assumptions will be made in Section 2, see also the monograph Bátkai and Piazzera [3]. In many cases, as explained above, it is easier to solve the equation without delay and the “pure” delay equation separately. In this case, it is natural to apply some operator splitting procedure described below.

Let us fix a time-step $h > 0$ and solve first the equation

$$(SP1/1) \quad \begin{cases} \frac{dv^{(1)}(t)}{dt} = \Phi v_t^{(1)}, & t \in [0, h], \\ v^{(1)}(0) = x =: x_1 \\ v_0^{(1)} = f =: f_1. \end{cases}$$

Then for $y_1 := v^{(1)}(h)$ we solve the equation

$$(SP1/2) \quad \begin{cases} \frac{dw^{(1)}(t)}{dt} = Bw^{(1)}(t), & t \in [0, h], \\ w^{(1)}(0) =: y_1. \end{cases}$$

To initialize the next step we set

$$\begin{aligned} x_2 &:= w^{(1)}(h) \\ f_2 &:= w_h^{(1)}. \end{aligned}$$

Then we start the procedure again by solving Equation (SP1) on the time-interval $t \in [h, 2h]$ with initial values $\begin{pmatrix} x_2 \\ f_2 \end{pmatrix}$. We iterate this procedure and solve the purely

delayed equation

$$(SPk/1) \quad \begin{cases} \frac{dv^{(k)}(t)}{dt} = \Phi v_t^{(k)}, & t \in [(k-1)h, kh], \\ v^{(k)}((k-1)h) = x_k, \\ v_{(k-1)h}^{(k)} = f_k. \end{cases}$$

Then we take $y_k := v^{(k)}(kh)$ and solve equation without delay

$$(SPk/2) \quad \begin{cases} \frac{dw^{(k)}(t)}{dt} = Bw^{(k)}(t), & t \in [(k-1)h, kh], \\ w^{(k)}((k-1)h) =: y_k. \end{cases}$$

Finally, we initialize

$$\begin{aligned} x_{k+1} &:= w^{(k)}(kh) \\ f_{k+1} &:= w_{kh}^{(k)}. \end{aligned}$$

The *sequentially split solution* at time level $t = kh$ is then

$$u^{sq}(kh) := x_{k+1}.$$

We shall show that for fixed $t \in [0, \infty)$ and for $h \rightarrow 0$ ($t = kh$, so $k \rightarrow \infty$) this split solution $u^{sq}(kh)$ converges to $u(t)$.

This procedure is especially useful, if we can drastically reduce the computational complexity of the problem by the splitting. This is, e.g., the case if $\Phi = C\delta(-1)$ is a point delay. Hence, we can integrate the first split equation explicitly, reducing the problem to solving the second equation, a classical partial differential equation. For a different splitting procedure, designed specifically for distributed delays, we refer to Csomós and Nickel [8].

Besides the sequential splitting there are other splitting methods known in the literature. However for the sake of simplicity, in this paper we only deal with the sequential and the first order Lie splitting. The convergence of higher order splitting schemes can be explained along the same lines. More refined convergence analysis of higher order splitting, and numerical experiments are subject to further research.

In Section 2, we show a way to rewrite a delay equation as an abstract Cauchy problem, and show how to associate nonlinear contraction semigroups to that. Also basic facts and about such semigroups are recalled there. These results are then used to explain the convergence of the splitting procedure. In the final Section 3 we study the order of convergence in some important special, linear cases.

2. THE DELAY SEMIGROUP

Consider the *abstract delay equation* of the following form (see, e.g., Bátkai and Piazzera [3]):

$$(1) \quad \begin{cases} \frac{du(t)}{dt} = Bu(t) + \Phi u_t, & t \geq 0, \\ u(0) = x \in H, \\ u_0 = f \in L^p([-1, 0]; H) \end{cases}$$

on the Hilbert space H where $1 \leq p < \infty$ and u_t is the history function, i.e.

$$u_t(\sigma) = u(t + \sigma), \quad t \geq 0, \sigma \in [-1, 0].$$

It is possible to transform the delay equation (1) into an abstract Cauchy problem, see Bátkai and Piazzera [3]. In order to do so, we take the product space $\mathcal{E} := H \times L^p([-1, 0]; H)$ and the new unknown function as

$$t \mapsto \mathcal{U}(t) := \begin{pmatrix} u(t) \\ u_t \end{pmatrix} \in \mathcal{E}.$$

Then (1) can be written as an abstract Cauchy problem on the space \mathcal{E}

$$(2) \quad \begin{cases} \frac{d\mathcal{U}(t)}{dt} = \mathcal{G}\mathcal{U}(t), & t \geq 0, \\ \mathcal{U}(0) = \begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{E}, \end{cases}$$

where the operator \mathcal{G} is given by the operator matrix

$$(3) \quad \mathcal{G} := \begin{pmatrix} B & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix}$$

on the domain

$$D(\mathcal{G}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(B) \times W^{1,p}([-1, 0]; H) : f(0) = x \right\}.$$

It is shown in Bátkai and Piazzera [3, Corollary 3.5, Proposition 3.9] that the delay equation (1) and the abstract Cauchy problem (2) are equivalent, i.e., they have the same solutions. More precisely, the first coordinate of the solution of (2) always solves (1).

Now the abstract Cauchy problem (2) can be solved by semigroup theoretic methods. These we recall briefly.

The best Lipschitz constant of a Lipschitz continuous function $T : X \rightarrow X$ on a Banach space X is denoted by $\|T\|_{\text{Lip}}$. The operator T is called then a *contraction* if $\|T\|_{\text{Lip}} \leq 1$. A subset $A \subset X \times X$ is called *dissipative* if for each $\alpha > 0$, $(I - \alpha A)^{-1}$ is a function and a contraction. We call A *m-dissipative* if $\overline{D(A)} = X$ and for all $\alpha > 0$ $\text{ran}(I - \alpha A) = X$ ($D(A)$ is the domain of definition of A). A family T of Lipschitz continuous operators on the Banach space X is called a *strongly continuous semigroup* of type $\omega \in \mathbb{R}$ if

- (1) $T(t) : X \rightarrow X$ for all $t \geq 0$,
- (2) $T(t)T(s)x = T(t+s)x$ for all $t, s \geq 0$, and $T(0) = I$,
- (3) $\|T(t)\|_{\text{Lip}} \leq e^{\omega t}$ for all $t \geq 0$,
- (4) $\lim_{t \rightarrow 0} T(t)x = x$.

The central point in this theory is the theorem of Crandall and Liggett [7] generalizing the classical Hille-Yosida Theorem.

Theorem 2.1. *Let $A \subset X \times X$ be ω -m-dissipative for some $\omega \in \mathbb{R}$, i.e. suppose that $A - \omega I$ is m-dissipative. Then A generates a strongly continuous semigroup of type ω given by the formula*

$$(4) \quad T(t)x = \lim_{n \rightarrow \infty} \left(I - \frac{t}{n} A \right)^{-n} x$$

for all $x \in X$ and $t \geq 0$.

If A generates a semigroup in this way, we call the abstract Cauchy problem associated to A well-posed on X . We return to our Cauchy problem (2), which under the next conditions becomes well-posed on \mathcal{E} .

Assumption 2.2. Suppose that

- (1) there is $\alpha \in \mathbb{R}$ such that $B - \alpha I$ is m-dissipative and hence B is the generator of a strongly continuous (nonlinear) semigroup V on H , and

- (2) $\Phi : W^{1,p}([-1, 0]; H) \rightarrow H$ is an operator defined as follows. Let $g : H \rightarrow H$ be a Lipschitz continuous function with Lipschitz constant β , and let $\eta : [-1, 0] \rightarrow \mathcal{L}(H)$ be a function of bounded variation with values as bounded linear operators. Assume further that

$$\eta(-1) = 0 \text{ and that } \lim_{\sigma \rightarrow -1} \eta(\sigma) \neq 0.$$

Now the operator Φ is given as

$$\Phi(f) := g \left(\int_{-1}^0 d\eta(\sigma) f(\sigma) \right) \quad \text{for } f \in C([-1, 0]; H).$$

Following Webb [25, Section 4], it is possible to find a new equivalent norm in \mathcal{E} such that the operator $\mathcal{G} - \gamma\mathcal{I}$ becomes an m -dissipative operator and hence the generator of a (nonlinear) semigroup. We define

$$\tau(r) := \int_{-1}^r |d\eta(\sigma)|$$

denote the total variation of η on $[-1, r]$. Let us introduce the new equivalent norm as

$$\left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\| := \left(\|x\|^p + \int_{-1}^0 |f(\sigma)|^p \tau(\sigma) d\sigma \right)^{\frac{1}{p}} \quad \text{for } \begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{E}.$$

Then we have the following result, see Webb [25, Proposition 4.1].

Theorem 2.3. *If $p > 1$, then $\mathcal{G} - \gamma\mathcal{I}$ is m -dissipative in \mathcal{E} for*

$$\gamma = \max\{0, \tau(0)(\frac{1}{p} + \frac{\beta^p}{q}) + \alpha\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. If $p = 1$ and $\beta \leq 1$, then $\mathcal{G} - \gamma\mathcal{I}$ is m -dissipative in \mathcal{E} , where $\gamma = \max\{0, \tau(0) + \alpha\}$.

Let us turn our attention to the splitting procedure described in the introduction. On the semigroup level in the product space \mathcal{E} this corresponds to the splitting

$$\mathcal{A}_1 = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix},$$

with

$$D(\mathcal{A}_1) = D(B) \times L^p([-1, 0], \tau; H)$$

and

$$D(\mathcal{A}_2) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in H \times W^{1,p}([-1, 0], \tau; H) : f(0) = x \right\}.$$

These operators generate strongly continuous semigroups of the form

$$(5) \quad \mathcal{T}_1(t) = \begin{pmatrix} V(t) & 0 \\ 0 & \tilde{I} \end{pmatrix}, \quad \mathcal{T}_2(t) \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} v(t) \\ v_t \end{pmatrix},$$

with \tilde{I} being the identity operator on $L^p([-1, 0], \tau; H)$ and

$$v(t) = x + \int_0^t \Phi v_\sigma d\sigma,$$

see Webb [25, Proposition 5.12]. If we denote the projection to the first coordinate with

$$\mathcal{P}_1 \begin{pmatrix} x \\ f \end{pmatrix} := x,$$

then the sequential splitting (after k steps with time step h) can be written as

$$u^{\text{sq}}(hk) = \mathcal{P}_1 \left[(\mathcal{T}_1(h)\mathcal{T}_2(h))^k \begin{pmatrix} x \\ f \end{pmatrix} \right].$$

Our first aim is to show the convergence $u^{\text{sq}}(hk) \rightarrow u(t) = \mathcal{P}_1 \mathcal{U}(t)$ for $k \rightarrow \infty$ and $kh = t$.

The main abstract technical tool in investigating splitting procedures are Lax-Chernoff type theorems and variants of the Lie-Trotter product formula for non-linear semigroups. Such result were proved in the paper by Brezis and Pazy [5, Section 3], and improved by Kobayashi [20]. We recall here the important results for our investigation.

Theorem 2.4. (a) Let A , B , and $\overline{A+B}$ are ω - m -dissipative sets in a Banach space X generating the semigroups S , T , and U , respectively. Then

$$U(t)x = \lim_{n \rightarrow \infty} \left[\left(I - \frac{t}{n} A \right)^{-1} \left(I - \frac{t}{n} B \right)^{-1} \right]^n x$$

for all $x \in X$.

(b) If, in addition, X is a Hilbert space and $A + B$ is closed, then

$$U(t)x = \lim_{n \rightarrow \infty} \left[S\left(\frac{t}{n}\right) T\left(\frac{t}{n}\right) \right]^n x.$$

We shall refer to the case (a) as the *Lie-splitting*, and to case (b) as the *sequential splitting*. For more general product formulae we refer to the seminal paper by Lions and Mercier [23].

The extra condition about the Banach space in (b) is important, as a counterexample by Kurtz and Pierre [22] shows. Nevertheless, one can fairly relax this condition by requiring that X has uniformly Gâteaux differentiable norm, see Kobayashi [21]. It seems to be folklore that a separable Banach space can be equivalently renormed such that the new norm has this differentiability property. Note however, that for a Hilbert space H and $1 < p < \infty$, the norm of the H -valued Bochner space $L^p(H)$ is uniformly Gâteaux differentiable (by a result of Day [9] the space $L^p(H)$ space is uniformly convex and uniformly smooth, which is even more than what is required; see also Diestel [10, Sec.2.3]). All in all we note that the above theorem is applicable in the situation of the delay equation, i.e., for the semigroups \mathcal{T}_1 and \mathcal{T}_2 on the product space $\mathcal{E} = H \times L^p([-1, 0]; H)$, as described above. Notice too that one has to endow the product space with an appropriate smooth and uniformly convex product norm (see Clarkson [6]).

After all these preparations, we obtain the following general convergence result.

Theorem 2.5. Suppose that Assumption 2.2 holds. Consider the delay equation (1) and the splitting of the operator $\mathcal{G} = \mathcal{A}_1 + \mathcal{A}_2$ described above. For every $p \in (1, \infty)$ the sequential splitting given by the formulae (SPk/1) and (SPk/2) and Lie splitting converge in $H \times L^p([-1, 0], \tau; H)$.

Proof. By Theorem 2.3 the semigroups \mathcal{T}_1 , \mathcal{T}_2 and \mathcal{T} are all of type γ for some $\gamma \in \mathbb{R}$. Since $\mathcal{A}_1 + \mathcal{A}_2$ is readily seen to be closed (and generates \mathcal{T}), Theorem 2.4 and the paragraph thereafter yield the proof. \square

3. SPLITTING IN LINEAR CASE

We now turn to the case of linear delay equations and prove convergence of the sequential splitting together with error estimates. Further analysis of more complicated equations is subject to ongoing research. For the terminology on linear semigroups we refer to Engel and Nagel [11].

First we show how an abstract semigroup result can be applied directly to the problem, demonstrating the power of the semigroup approach. To obtain some error estimates one can apply a general result by Hansen and Ostermann [16, Theorem 3.1].

Theorem 3.1. *Assume that operators \mathcal{A}_1 and \mathcal{A}_2 generate (linear) contraction C_0 -semigroups in the Hilbert space \mathcal{E} and that $\mathcal{G} = \overline{\mathcal{A}_1 + \mathcal{A}_2}$ generates the strongly continuous semigroup \mathcal{T} and satisfies*

$$(6) \quad D(\mathcal{G}^2) \subset D(\mathcal{A}_1 \mathcal{A}_2).$$

Then the for sufficiently smooth initial values the Lie- and the sequential splittings have first order convergence, i.e., for every $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{G}^2)$,

$$\begin{aligned} \left\| (I - h\mathcal{A}_1)^{-1} (I - h\mathcal{A}_2)^{-1} \begin{pmatrix} x \\ f \end{pmatrix} - \mathcal{T}(h) \begin{pmatrix} x \\ f \end{pmatrix} \right\| &\leq C \cdot h^2 \cdot \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{\mathcal{G}^2} \quad \text{and} \\ \left\| \mathcal{T}_1(h) \mathcal{T}_2(h) \begin{pmatrix} x \\ f \end{pmatrix} - \mathcal{T}(h) \begin{pmatrix} x \\ f \end{pmatrix} \right\| &\leq C \cdot h^2 \cdot \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{\mathcal{G}^2}. \end{aligned}$$

For other abstract error estimates, see for example Jahnke and Lubich [19] or Faragó and Havasi [12].

Theorem 3.2. *Assume that B generates a linear contraction C_0 -semigroup and that the linear operator Φ satisfies the condition $\text{ran } \Phi \subset D(B)$. Then for $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{G}^2)$ the sequential splitting is of first order.*

Proof. To prove the desired estimate, we have to check the conditions in Theorem 3.1 for the operators

$$\mathcal{A}_1 = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix},$$

with their respective domains, see Section 2.

By Theorem 2.3, we know that each operator \mathcal{A}_i generates a contraction semigroup in \mathcal{E} . Hence, we only have to check the domain condition.

Consider first the right-hand side. Then

$$\begin{aligned} D(\mathcal{A}_1 \mathcal{A}_2) &= \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}_2) : \mathcal{A}_2 \begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}_1) \right\} \\ &= \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}_2) : \Phi f \in D(B) \right\} = D(\mathcal{A}_2) \end{aligned}$$

since $\text{ran } \Phi \subset D(B)$.

For the left-hand side we obtain

$$\begin{aligned} D(\mathcal{G}^2) &= \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{G}) : \mathcal{G} \begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{G}) \right\} = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{G}) : \begin{pmatrix} Bx + \Phi f \\ f' \end{pmatrix} \in D(\mathcal{G}) \right\} = \\ &= \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(B) \times W^{2,p}([-1, 0], \tau; H) : f(0) = x, f'(0) = Bx + \Phi f \in D(B) \right\}. \end{aligned}$$

Hence,

$$D(\mathcal{G}^2) \subset D(\mathcal{G}) \subset D(\mathcal{A}_2) = D(\mathcal{A}_1 \mathcal{A}_2),$$

and the assertion is proved. \square

We now want to weaken the assumption about $\text{ran } \Phi$ and still obtain a convergence rate result. To do so we directly calculate the local error of the splitting. The splitting procedure is certainly stable, as all appearing semigroups are of type γ . So without changing the stability of the splitting, we can take arbitrary equivalent norms on \mathcal{E} , and so get rid of the function τ . Let us record this fact for later reference: there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$(7) \quad \|(\mathcal{T}_1(t) \mathcal{T}_2(t))^k\| \leq M e^{\omega k t} \quad \text{holds for all } t \geq 0, k \in \mathbb{N},$$

where \mathcal{T}_1 and \mathcal{T}_2 are the semigroups generated by the operators

$$\mathcal{A}_1 = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix}$$

with $D(\mathcal{A}_1) = D(B) \times L^p([-1, 0]; H)$ and

$$D(\mathcal{A}_2) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in H \times W^{1,p}([-1, 0]; H) : f(0) = x \right\},$$

and are given by (5).

Further, by the calculations in Section 2 and by the results in Bátkai and Piazzera [3, Section 3.1], we know that the semigroup \mathcal{T} generated by $(\mathcal{G}, D(\mathcal{G}))$ is given by

$$(8) \quad \mathcal{T}(t) \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} u(t) \\ u_t \end{pmatrix},$$

and we have

$$(9) \quad u_t(\sigma) = \begin{cases} u(t + \sigma) & \text{if } t + \sigma > 0, \\ f(t + \sigma) & \text{if } t + \sigma \leq 0. \end{cases}$$

Here is now the assumption replacing the one “ $\text{ran } \Phi \subseteq D(B)$ ”. The price we shall pay for weakening the assumption is, however, that we will have to improve the regularity of the initial condition.

Assumption 3.3. Let Y be a Banach space, $p \in (1, \infty)$, and consider the linear operator

$$\Phi : W^{1,p}([-1, 0]; Y) \rightarrow Y$$

given by

$$\Phi f = \int_{-1}^0 d\eta(\sigma) f(\sigma),$$

with $\eta : [0, 1] \rightarrow \mathcal{L}(Y)$ of bounded variation, such that for some $\sigma_0 < 0$

$$\eta : [\sigma_0, 0] \rightarrow \mathcal{L}(Y) \quad \text{is Lipschitz continuous.}$$

- a) Suppose that the above holds for $Y = H$.
- b) Suppose that the above holds for $Y = D(B)$.
- c) Assume further that

$$\eta(-1) = 0 \text{ and that } \lim_{\sigma \rightarrow -1} \eta(\sigma) \neq 0.$$

Informally, this means that Φ maps $D(B)$ -valued functions into $D(B)$ and that there are no point delays accumulating around zero. Note that if the operator Φ is “scalar”, for example if $\Phi = c\delta_{-1}$ with $c \in \mathbb{C}$, then this condition is always satisfied, while the conditions of the previous Theorem 3.2 are not.

In what follows, we abbreviate the norm notation $\|\cdot\|_{L^p([-1, 0]; D(B))}$ to $\|\cdot\|_{L^p(D(B))}$, and similarly to all function spaces defined on $[-1, 0]$ with values in H or $D(B)$.

Theorem 3.4. Assume that $p \geq 1$, B generates a linear contraction semigroup in the Hilbert space H , and that Assumption 3.3 a) and b) are satisfied. Define

$$\mathcal{D} := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(B^2) \times W^{1,p}([-1, 0]; D(B)), f \in \text{Lip}([-1, 0]; H), f(0) = x \right\}.$$

Then for $\begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{D}$ the sequential splitting is of first order. More precisely, for all $T_0 > 0$ there is a constant $C > 0$ such that

$$\left\| \left(\left(\mathcal{T}_1\left(\frac{t}{n}\right) \mathcal{T}_2\left(\frac{t}{n}\right) \right)^n - \mathcal{T}(t) \right) \begin{pmatrix} x \\ f \end{pmatrix} \right\| \leq \frac{C}{n} \left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{\mathcal{D}} \quad \text{for all } t \in [0, T_0],$$

where for $\binom{x}{f} \in \mathcal{D}$ one defines

$$\left\| \binom{x}{f} \right\|_{\mathcal{D}} := \|x\|_B + \|Bx\|_B + \|f\|_{W^{1,p}(D(B))} + \|f\|_{\text{Lip}(H)}.$$

The proof of this theorem follows the standard route of stability analysis and estimating the local error using the variation of constants formula.

These are carried out in the series of lemmas

where we first show the invariance of the subspace \mathcal{D} under the delay semigroup, and then show the local error estimate for initial values from this set.

As before, denote by $\mathcal{P}_1 : \mathcal{E} \rightarrow H$ and $\mathcal{P}_2 : \mathcal{E} \rightarrow L^p([-1, 0]; H)$ the coordinate projections.

Lemma 3.5. *Let $\binom{x}{f} \in D(\mathcal{G})$ be such that $f \in \text{Lip}([-1, 0]; H)$ with Lipschitz constant L . Then for all $t \in [0, 1]$ the function $u_t := \mathcal{P}_2 \mathcal{T}(t) \binom{x}{f}$ is Lipschitz continuous with constant*

$$\max \left\{ L, M \left\| \binom{x}{f} \right\|_{\mathcal{G}} \right\},$$

with M depending only on \mathcal{T} .

Proof. By (9) we know that

$$u_t : [-t, 0] \rightarrow H, \quad u_t(\sigma) = u(t + \sigma).$$

This is a Lipschitz continuous function, since it is continuously differentiable, with constant

$$\sup_{\sigma \in [-t, 0]} \|\mathcal{P}_1 \mathcal{T}(t + \sigma) \mathcal{G} \binom{x}{f}\| \leq M \|\mathcal{G} \binom{x}{f}\|.$$

Now, since f is assumed to be Lipschitz continuous and since $f(0) = x = u_t(-t)$ (for $\binom{x}{f} \in D(\mathcal{G})$), and also by taking into account (9) again, it follows that u_t is Lipschitz continuous with the asserted constant. \square

Lemma 3.6. *Let $\binom{x}{f} \in \mathcal{E}$, let $v_t := \mathcal{P}_2(\mathcal{T}_2(t) \binom{x}{f})$ and $u_t := \mathcal{P}_2(\mathcal{T}(t) \binom{x}{f})$. Then for all $t \in [0, -\sigma_0]$ we have*

$$\Phi v_t = \Phi u_t + \mathcal{O}(t),$$

where $\mathcal{O}(t)$ denotes a term of magnitude $t \cdot C \max\{\|f\|_{L^p(H)}, \|x\|\}$ with a constant C independent of x and f .

Proof. Similarly to (9) we have

$$v_t(\sigma) = \begin{cases} v(t + \sigma) & \text{if } t + \sigma > 0, \\ f(t + \sigma) & \text{if } t + \sigma \leq 0. \end{cases}$$

We now can write

$$\begin{aligned} \|\Phi u_t - \Phi v_t\| &= \left\| \int_{-1}^0 d\eta(\sigma) (u_t(\sigma) - v_t(\sigma)) \right\| \\ &\leq \int_{-1}^{-t} \|u_t(\sigma) - v_t(\sigma)\| \|d\eta(\sigma)\| + \int_{-t}^0 \|u(t + \sigma) - v(t + \sigma)\| \|d\eta(\sigma)\| \\ &\leq \int_{-1}^{-t} \|f(t + \sigma) - f(t + \sigma)\| \|d\eta(\sigma)\| + \int_{-t}^0 \|u(t + \sigma) - v(t + \sigma)\| \|d\eta(\sigma)\| \\ &= 0 + \int_{-t}^0 \|u(t + \sigma) - v(t + \sigma)\| \|d\eta(\sigma)\| \leq 2M \max\{\|f\|_{L^p(H)}, \|x\|\} t, \end{aligned}$$

where M depends on the bounds of the semigroups \mathcal{T} , \mathcal{T}_2 and on the Lipschitz constant of η . \square

Lemma 3.7. *a) The subspace \mathcal{D} is invariant under the semigroup \mathcal{T} .*

b) For $\begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{D}$ the function

$$[0, -\sigma_0] \ni s \mapsto \Phi u_s \in H$$

is Lipschitz continuous with constant $C(\|x\|_B + \|f\|_{\text{Lip}(H)} + \|f\|_{W^{1,p}(H)})$.

c) For $\begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{D}$ the function

$$[0, -\sigma_0] \ni s \mapsto \Phi u_s \in D(B)$$

is bounded in norm, and the function

$$[0, -\sigma_0] \ni s \mapsto V(\sigma)\Phi u_s \in H$$

is Lipschitz continuous with constant $C(\|x\|_B + \|f\|_{W^{1,p}(D(B))} + \|f\|_{\text{Lip}(H)} + \|Bx\|_B)$.

Proof. a) Let $\tilde{\mathcal{T}}$ be the semigroup generated by the part $\tilde{\mathcal{G}}$ of \mathcal{G} in $\tilde{\mathcal{E}} := D(B) \times L^p([-1, 0]; D(B))$. This semigroup exists by the general facts presented in the beginning of Section 2 and by Assumption 3.3.b). The domain of the generator is

$$D(\tilde{\mathcal{G}}) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(B^2) \times W^{1,p}([-1, 0]; D(B)), f(0) = x \right\}.$$

Hence

$$\mathcal{D} = D(\tilde{\mathcal{G}}) \cap (H \times \text{Lip}([-1, 0]; H)).$$

The space $D(\tilde{\mathcal{G}})$ is invariant under $\tilde{\mathcal{T}}$. It is easy to see that for $\begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{D}$ we have $\mathcal{T}(t)\begin{pmatrix} x \\ f \end{pmatrix} = \tilde{\mathcal{T}}(t)\begin{pmatrix} x \\ f \end{pmatrix}$ for all $t \geq 0$. This follows from the fact that $\mathcal{D} \subset D(\mathcal{G})$, from the uniqueness of the classical solutions of (1), and from the comparison of the topologies of \mathcal{E} and $\tilde{\mathcal{E}}$. Next, since $D(\tilde{\mathcal{G}}) \subseteq D(\mathcal{G})$ from Lemma 3.5 we obtain that $\mathcal{T}(t)\begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{D}$.

b) Let $0 \leq s \leq t \leq -\sigma_0$ and take $\begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{D}$. By the definition of u_t we have

$$\begin{aligned} \|\Phi u_s - \Phi u_t\| &= \left\| \int_{-1}^0 d\eta(\sigma) (u_s(\sigma) - u_t(\sigma)) \right\| \leq \int_{-1}^0 \|u_s(\sigma) - u_t(\sigma)\| \|d\eta(\sigma)\| \\ &= \int_{-1}^{-t} \|f(s + \sigma) - f(t + \sigma)\| \cdot \|d\eta(\sigma)\| \\ &\quad + \int_{-t}^{-s} \|f(s + \sigma) - u(t + \sigma)\| \cdot \|d\eta(\sigma)\| \\ &\quad + \int_{-s}^0 \|u(s + \sigma) - u(t + \sigma)\| \cdot \|d\eta(\sigma)\|. \end{aligned}$$

The first of these three terms can be estimated as

$$\int_{-1}^{-t} \|f(s + \sigma) - f(t + \sigma)\| \cdot \|d\eta(\sigma)\| \leq \|f\|_{\text{Lip}(H)} \|\eta\|_{\text{BV}(\mathcal{L}(H))} \cdot (t - s).$$

Similarly, since u is a classical solution of the delay equation (by $\mathcal{D} \subset D(\tilde{\mathcal{G}}) \subset D(\mathcal{G})$), it is continuously differentiable and hence Lipschitz continuous. Hence the

third term can be bounded in a similar way as the first. For the Lipschitz constant here note that

$$\binom{u'(t)}{u'_t} = \tilde{\mathcal{T}}(t)\tilde{\mathcal{G}}\binom{x}{f} = \mathcal{T}(t)\mathcal{G}\binom{x}{f},$$

and hence

$$L \leq M(\|Bx + \Phi f\| + \|f\|_{W^{1,p}(H)}).$$

Finally, in the second term we integrate on the interval $[-t, -s]$ with $t, s \leq -\sigma_0$. Hence this term can be bounded by $C(t-s)(\|x\| + \|f\|_{L^p(H)})$, where C depends on \mathcal{T} and on the Lipschitz constant of η on $[\sigma_0, 0]$.

c) By the considerations in part a), we have that for $\binom{x}{f} \in \mathcal{D}$, the function

$$s \mapsto \mathcal{P}_2 \tilde{\mathcal{T}}(s) \binom{x}{f} \in W^{1,p}([-1, 0]; D(B))$$

is continuous, and hence especially bounded on $[0, 1]$ by

$$\sup_{s \in [0, 1]} \|\mathcal{P}_2 \tilde{\mathcal{T}}(s) \binom{x}{f}\|_{W^{1,p}(D(B))} \leq \sup_{s \in [0, 1]} \|\tilde{\mathcal{T}}(s) \binom{x}{f}\|_{\tilde{\mathcal{G}}} \leq M(\|Bx\|_B + \|f\|_{W^{1,p}(D(B))}).$$

This shows the first assertion, since $\Phi \in \mathcal{L}(W^{1,p}([-1, 0]; D(B)), D(B))$ by Assumption 3.3.b). The second statement is a trivial consequence now, since for large $\lambda > 0$ the function $[0, 1] \ni s \mapsto V(s)R(\lambda, B) \in \mathcal{L}(H)$ is Lipschitz, hence so is $s \mapsto V(s)R(\lambda, B)(\lambda - B)\Phi u_s$ if we take into account also part b). The Lipschitz constant is

$$L = C(\|x\|_B + \|f\|_{W^{1,p}(D(B))} + \|f\|_{\text{Lip}(H)} + \|Bx + \Phi f\|_B),$$

where C does not depend on x and f . \square

Lemma 3.8 (Local error). *There is a constant $K > 0$ such that for all $t \in [0, -\sigma_0]$ we have*

$$\left\| \mathcal{T}_1(t)\mathcal{T}_2(t) \binom{x}{f} - \mathcal{T}(t) \binom{x}{f} \right\|_{\mathcal{E}} \leq Kt^2 \left\| \binom{x}{f} \right\|_{\mathcal{D}} \quad \text{for all } \binom{x}{f} \in \mathcal{D}.$$

Proof. Take $\binom{x}{f} \in \mathcal{D}$, and let u, u_t, v, v_t be as in (5) and (8), respectively. We can write

$$V(t)v(t) - u(t) = V(t)x + V(t) \int_0^t \Phi v_s ds - V(t)x - \int_0^t V(t-s)\Phi u_s ds.$$

By Lemma 3.6, this is further equal to

$$\begin{aligned} &= V(t)x + V(t) \int_0^t \Phi u_s ds - V(t)x - \int_0^t V(t-s)\Phi u_s ds + \int_0^t \mathcal{O}(s) ds \\ &= \int_0^t V(t-s)(V(s)\Phi u_s - \Phi v_s) ds + \mathcal{O}(t^2) \\ &= \int_0^t V(t-s)(V(s)\Phi u_s - \Phi u_0) ds + \int_0^t V(t-s)(\Phi u_0 - \Phi u_s) ds + \mathcal{O}(t^2). \end{aligned}$$

By Lemma 3.7, the function $[0, -\sigma_0] \ni s \mapsto \Phi u_s$ is Lipschitz-continuous, and the function $[0, -\sigma_0] \ni s \mapsto V(s)\Phi u_s$ is Lipschitz continuous. Denoting their Lipschitz constant with L , we obtain that

$$\|V(t)u(t) - v(t)\| \leq MLt^2 + MLt^2 + \|\mathcal{O}(t^2)\| \leq Ct^2 \left\| \binom{x}{f} \right\|_{\mathcal{D}}.$$

As for the second coordinates, we can write

$$\begin{aligned}\|u_t - v_t\|_{L^p(H)}^p &= \int_{-1}^0 \|u_t(\sigma) - v_t(\sigma)\|^p d\sigma \\ &= \int_{-t}^0 \|u(t+\sigma) - v(t+\sigma)\|^p d\sigma \leq Ct^{2p+1}(\|f\|_{L^p(H)} + \|x\|)^p.\end{aligned}$$

This finishes the proof by defining K in terms of the constant C from the above. \square

Proof of Theorem 3.4. Without loss of generality we may assume $T_0 = 1$ (the constant C shall depend on T_0 , however). By a standard telescopic summation we get

$$(10) \quad \left(\mathcal{T}_1\left(\frac{t}{n}\right)\mathcal{T}_2\left(\frac{t}{n}\right)\right)^n - \mathcal{T}(t) = \sum_{k=0}^{n-1} \left(\mathcal{T}_1\left(\frac{t}{n}\right)\mathcal{T}_2\left(\frac{t}{n}\right)\right)^{n-k} \left(\mathcal{T}_1\left(\frac{t}{n}\right)\mathcal{T}_2\left(\frac{t}{n}\right) - \mathcal{T}\left(\frac{t}{n}\right)\right) \mathcal{T}\left(\frac{kt}{n}\right).$$

As we have seen in Lemma 3.7, the set \mathcal{D} is invariant under the delay semigroup \mathcal{T} , and also from this Lemma 3.7 and Lemma 3.5 it follows that for some $M \geq 1$

$$\left\|\mathcal{T}(s)\begin{pmatrix} x \\ f \end{pmatrix}\right\|_{\mathcal{D}} \leq M \left\|\begin{pmatrix} x \\ f \end{pmatrix}\right\|_{\mathcal{D}} \quad \text{for all } s \in [0, 1], \begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{D}.$$

From Theorem 2.5 we obtain a possibly larger M , which makes it possible to estimate the first factors on the right hand side of (10) by exponential terms. Hence, by applying Lemma 3.8 we obtain that

$$\left\|\left(\left(\mathcal{T}_1\left(\frac{t}{n}\right)\mathcal{T}_2\left(\frac{t}{n}\right)\right)^n - \mathcal{T}(t)\right)\begin{pmatrix} x \\ f \end{pmatrix}\right\|_{\mathcal{E}} \leq \sum_{k=0}^{n-1} M e^{\omega kt/n} K \left(\frac{t}{n}\right)^2 \left\|\mathcal{T}\left(\frac{kt}{n}\right)\begin{pmatrix} x \\ f \end{pmatrix}\right\|_{\mathcal{D}} \leq \frac{C}{n} \left\|\begin{pmatrix} x \\ f \end{pmatrix}\right\|_{\mathcal{D}},$$

where $C = M^2 K e^{|\omega|}$. \square

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A.B., EÖTVÖS LORÁND UNIVERSITY, INSTITUTE OF MATHEMATICS, 1117 BUDAPEST, PÁZMÁNY P. SÉTÁNY 1/C, HUNGARY.

E-mail address: `batka@cs.elte.hu`

P.Cs., UNIVERSITÄT INNSBRUCK, INSTITUT FÜR MATHEMATIK, TECHNIKERSTRASSE 13, 6020 INNSBRUCK, AUSTRIA

E-mail address: `petra.csomos@uibk.ac.at`

B.F., TECHNISCHE UNIVERSITÄT DARMSTADT, FACHBEREICH MATHEMATIK, SCHLOSSGARTENSTR. 7, 64289 DARMSTADT, GERMANY

E-mail address: `farkas@mathematik.tu-darmstadt.de`